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A Block Negacyclic Bush-Type Hadamard Matrix and Two Strongly Regular Graphs

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A block negacyclic Bush-type Hadamard matrix of order 36 is used in a symmetric $BGW(26, 25, 24)$ with zero diagonal over a cyclic group of order 12 to construct a twin strongly regular graph with parameters $v = 936$, $k = 375$, $\lambda = \mu = 150$ whose points can be partitioned in 26 cocliques of size 36. The same Hadamard matrix then is used in a symmetric $BGW(50, 49, 48)$ with zero diagonal over a cyclic group of order 12 to construct a Siamese twin strongly regular graph with parameters $v = 1800$, $k = 1029$, $\lambda = \mu = 588$.

Key Words: symmetric design; Bush-type Hadamard matrix; strongly regular graph; block negacyclic matrix; balanced generalized weighing matrix.

1. INTRODUCTION

A *symmetric* 2-(v, k, λ) *design* is a set X of v points together with a collection B of v k-subsets of X called blocks such that every two points are contained in exactly λ blocks. A symmetric 2-(v, k, λ) design can be described in terms of its incidence matrix, being a square v by v (0, 1)-matrix with constant row and column sum equal to k, and constant scalar product of pairs of rows equal to λ. For more on designs see Beth *et al.* [1].

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A *Hadamard matrix* of order n is a square n by n matrix with entries ± 1 whose rows are pairwise orthogonal. A Hadamard matrix H of order $4n^2$ is called *regular* if every row and column sum of H is $2n$.

A *Bush-type* Hadamard matrix [4] is a regular Hadamard matrix of order $4n^2$ with the additional property of being a block matrix $H = [H_{ij}]$, with blocks of size $2n$ such that $H_{ii} = J_{2n}$ and $H_{ij} J_{2n} = J_{2n} H_{ij} = 0$, $i \neq j$, $1 \leqslant i \leqslant 2n$, $1 \leqslant j \leqslant 2n$, where J_m denotes the all-one m by m matrix.

Let $U = \mathrm{circ}(0, 1, 0, \ldots, 0)$ be the circulant matrix of order n with first row $(010\ldots0)$ and $N = \mathrm{diag}(-1, 1, 1, \ldots, 1)$ be the diagonal matrix of order n with -1 at the $(1, 1)$-position and 1 elsewhere on the diagonal. Let $\Omega = UN$. Matrices which are polynomials in Ω are called *negacyclic*; see [5, 2]. A block negacyclic Bush-type Hadamard matrix of order $4n^2$ is a Bush-type Hadamard matrix which is block negacyclic (with block size $2n$, of course) with all blocks being symmetric.

Bush [4] showed that the existence of a projective plane of order $2n$ implies the existence of a symmetric Bush-type Hadamard matrix of order $4n^2$. A result of Kharaghani [11] can be used to show that the existence of a Hadamard matrix of order $4n$ implies the existence of a negacyclic Bush-type Hadamard matrix of order $16n^2 = (4n)^2$. No examples of block negacyclic Bush-type Hadamard matrices of order $4n^2$, $n > 1$ n odd, appeared to be known previously.

A *strongly regular graph* with parameters (v, k, λ, μ), denoted $SRG(v, k, \lambda, \mu)$, is a regular graph with v vertices of degree k such that every two adjacent vertices have exactly λ common neighbors, and every two nonadjacent vertices have exactly μ common neighbors; see [3].

A *balanced generalized weighing matrix* $BGW(v, \kappa, \lambda)$ over a multiplicative group G is a v by v matrix $W = [g_{ij}]$ with entries from $\bar{G} = G \cup \{0\}$ such that each row of W contains exactly κ nonzero entries, and for every $a, b \in \{1, \ldots, v\}$, $a \neq b$, the multi-set

$$\{g_{ai} g_{bi}^{-1} : 1 \leqslant i \leqslant v,\ g_{ai} \neq 0,\ g_{bi} \neq 0\}$$

contains exactly $\lambda / |G|$ copies of each element of G.

In a recent work [9], Kharaghani introduced a method in which from a single Bush-type Hadamard matrix of order $4n^2$ one can generate an infinite class of twin symmetric designs provided that one of the numbers $2n-1$ or $2n+1$ is a prime power, and two infinite classes of twin symmetric designs if both $2n-1$ and $2n+1$ are prime powers. A twin design is a $(0, \pm 1)$-matrix where both signs, 1 and -1, form symmetric designs with the same parameters.

In this paper, for the first time, a block negacyclic Bush-type Hadamard matrix of order 36 is constructed and used in a symmetric $BGW(26, 25, 24)$ with zero diagonal over the cyclic group of order 12 to generate a twin

$SRG(936, 375, 150, 150)$ whose vertices can be partitioned into 26 cocliques of size 36. Using the same matrix, but in a symmetric $BGW(50, 49, 48)$ with zero diagonal over the cyclic group of order 12, we are also able to generate a Siamese twin $SRG(1800, 1029, 588, 588)$. A Siamese twin SRG is a symmetric $(0, \pm 1)$-matrix with zero diagonal for which both signs, 1 and -1 to which a certain number of negated 1 entries are added, are the incidence matrices of strongly regular graphs with the same parameters sharing cliques of the same size; see [10] for the definition of Siamese twin designs.

2. A BLOCK NEGACYCLIC BUSH-TYPE HADAMARD MATRIX OF ORDER 36

In this section we first introduce the first block negacyclic Bush-type Hadamard matrix of order 36. This is a Hadamard matrix H of order 36 of the following form:

$$H = \begin{pmatrix} A & B & C & D & E & F \\ -F & A & B & C & D & E \\ -E & -F & A & B & C & D \\ -D & -E & -F & A & B & C \\ -C & -D & -E & -F & A & B \\ -B & -C & -D & -E & -F & A \end{pmatrix},$$

where the symmetric blocks are

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 & - & - & - \\ 1 & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 \\ - & - & 1 & 1 & 1 & - \\ - & 1 & - & 1 & 1 & - \\ - & 1 & 1 & - & - & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} - & 1 & - & 1 & - & 1 \\ 1 & - & - & - & 1 & 1 \\ - & - & 1 & 1 & - & 1 \\ 1 & - & 1 & - & 1 & - \\ - & 1 & - & 1 & 1 & - \\ 1 & 1 & 1 & - & - & - \end{pmatrix}$$

$$D = \begin{pmatrix} - & - & 1 & - & 1 & 1 \\ - & 1 & 1 & - & 1 & - \\ 1 & 1 & - & 1 & - & - \\ - & - & 1 & 1 & - & 1 \\ 1 & 1 & - & - & - & 1 \\ 1 & - & - & 1 & 1 & - \end{pmatrix}$$

$$E = \begin{pmatrix} - & 1 & - & - & 1 & 1 \\ 1 & - & 1 & - & - & 1 \\ - & 1 & 1 & - & 1 & - \\ - & - & - & 1 & 1 & 1 \\ 1 & - & 1 & 1 & - & - \\ 1 & 1 & - & 1 & - & - \end{pmatrix}$$

$$F = \begin{pmatrix} - & 1 & - & 1 & - & 1 \\ 1 & 1 & - & - & 1 & - \\ - & - & 1 & 1 & 1 & - \\ 1 & - & 1 & 1 & - & - \\ - & 1 & 1 & - & - & 1 \\ 1 & - & - & - & 1 & 1 \end{pmatrix}.$$

To explain our method of search for this matrix briefly, we denote the rows of H by $l1, l2, \ldots, l36$. The jth entry in the row li is denoted by $l(i, j)$. At the beginning the following reduction is essential. Permuting the rows and columns of H passing through the first block B (in the first block-row) we may assume (without loss of generality) that

$$l(1, 7) = 1,\ l(1, 8) = 1,\ l(1, 9) = 1,\ l(1, 10) = -1,\ l(1, 11) = -1,\ l(1, 12) = -1.$$

Note that in every block-matrix we have in each row (and column) 3 times the entry 1 and 3 times the entry -1. Therefore (for example) for the entries $l(1, 13)$, $(l1, 14)$, $l(1, 15)$, $l(1, 16)$, $l(1, 17)$, $l(1, 18)$ we have exactly 20 possibilities. Therefore for the first row $l1$ of H we have at the start 20^4 possibilities. It is clear that it is enough to construct only the first 6 rows $l1$, $l2$, $l3$, $l4$, $l5$, $l6$.

Now, when the row $l1$ is chosen, then the rows $l7$, $l13$, $l19$, $l25$ and $l31$ are uniquely determined. But all these rows have to be pair-wise orthogonal. This diminishes the number of possibilities for $l1$ and in fact we get exactly 13268 possibilities for $l1$.

Similarly , for the row $l2$ we have at the start exactly 20^5 possibilities (no more reduction!). When $l2$ is chosen, then the rows $l8$, $l14$, $l20$, $l26$, $l32$ are uniquely determined and all these rows (and the previously constructed) should be pair-wise orthogonal. Also we should not forget that all block matrices must be symmetric and this gives additional conditions which diminish the number of possibilities for $l1$ and $l2$. As a result we get exactly 53268 possibilities for $l1$ and $l2$.

We have arrived at the critical point of the construction because of the explosion of possibilities! Similarly, we start with the construction of $l3$. Here the computer computation takes 2 to 3 days and we get exactly 576 possibilities for $l1$, $l2$, $l3$. This is already half of the matrix H. From now on the construction goes quickly: for $l1$, $l2$, $l3$, $l4$ we get exactly 320 possibilities; for $l1$, $l2$, $l3$, $l4$, $l5$ we get exactly 288 possibilities and finally for $l1$, $l2$, $l3$, $l4$, $l5$, $l6$ we get exactly (also) 288 possibilities! The search is successful and the matrix H above is constructed. The symmetric $(36, 15, 6)$-design obtained from the matrix H is rigid and, unfortunately, a search of this sort seems not feasible for the next interesting negacyclic Bush-type Hadamard matrix which has to be of order 100.

The matrix H being block negacyclic commutes with block negacyclic matrices of the same order (and the same block size, of course), if the blocks all commute and this is essential for our method of construction for strongly regular graphs in this paper.

3. SOME SYMMETRIC BALANCED GENERALIZED WEIGHING MATRICES WITH ZERO DIAGONAL

Despite the fact that several method of construction is available for balanced generalized matrices, see, for example, [6, 8], not much is known about the existence of symmetric or skew balanced generalized weighing matrices with zero diagonal. Symmetric *BGW*s with zero diagonal, as we will see, are essential to our construction of *SRG*s.

In this section we reproduce some known construction methods to obtain a symmetric $BGW(26, 25, 24)$ and $BGW(50, 49, 48)$ with zero diagonal over the cyclic group C_{12} of order 12. Our main reference for the construction of *BGW*s is Gibbons and Mathon [6].

LEMMA 1. *There are symmetric* $BGW(26, 25, 24)$ *and* $BGW(50, 49, 48)$ *with zero diagonal over the cyclic group* C_{12} *of order* 12.

Proof. Let $25=(2)(12)+1$, $GF(25)=\{a_1, a_2, \ldots, a_{25}\}$ and α a primitive root in $GF(25)$. Note that if a_k-a_j belongs to the cyclotomic class $\{\alpha^i, \alpha^{12+i}\}$ for some i, then a_j-a_k also belongs to the same cyclotomic class. Now apply the cyclotomic method of construction given by Gibbons and Mathon in [6], page 8. The same argument applies to the prime power 7^2.

Remark 2. Note that the cyclotomic method of construction for the BGW matrices given in [6] always provides a skew matrix for $BGW(q+1, q, q-1)$ over the cyclic group C_{q-1} of order $q-1$, if q is an odd prime power.

4. TWO TWIN STRONGLY REGULAR GRAPHS

We now have the building blocks for the two twin strongly regular graphs $SRG(936, 375, 150, 150)$ and $SRG(1800, 1029, 588, 588)$.

Let G_{4n} be the cyclic subgroup of order $4n$ of signed permutation matrices generated by the negacyclic matrix Ω, defined in Section 1, of order $2n$. Then $C_{4n}=\{G_{4n}\otimes I_{2n}\}$ is a cyclic subgroup (of order $4n$) of the signed permutation matrices of order $4n^2$. The next lemma is crucial for our construction.

LEMMA 3. *Let Γ be any element in C_{4n}, H a block negacyclic matrix of order $4n^2$ and $L=R_{2n}\otimes I_{2n}$, where I_{2n} is the identity matrix of order $2n$ and R_{2n} denotes the back diagonal identity matrix of order $2n$ (for simplicity we will drop the indices where there is no fear of confusion). Then the matrix $H\Gamma L$ is symmetric.*

Proof. Let $(A_1, A_2, \ldots, A_{2n})$ be the first row (blocks) of H. So we can write

$$H=I\otimes A_1+\Omega\otimes A_2+\cdots+\Omega^{2n-1}\otimes A_{2n},$$

where Ω is the negacyclic shift matrix of order $2n$ defined in Secion 1. Let $\Gamma=\Omega^i\otimes I$, for some i, $0\leqslant i\leqslant 4n$. Now,

$$\begin{aligned}
(H\Gamma L)^t &= L\Gamma^t H^t\\
&=(R\otimes I)((\Omega^i)^t\otimes I)(I\otimes A_1^t+\Omega^t\otimes A_2^t+\cdots+(\Omega^{2n-1})^t\otimes A_{2n-1}^t)\\
&=(R(\Omega^i)^t\otimes I)(I\otimes A_1+\Omega^t\otimes A_2+\cdots+(\Omega^{2n-1})^t\otimes A_{2n-1})\\
&=((\Omega^i)\otimes I)(R\otimes I)(I\otimes A_1+\Omega^t\otimes A_2+\cdots+(\Omega^{2n-1})^t\otimes A_{2n-1})\\
&=((\Omega^i)\otimes I)(R\otimes A_1+R\Omega^t\otimes A_2+\cdots+R(\Omega^{2n-1})^t\otimes A_{2n-1})\\
&=((\Omega^i)\otimes I)(I\otimes A_1+\Omega\otimes A_2+\cdots+\Omega^{2n-1}\otimes A_{2n-1})(R\otimes I)\\
&=(I\otimes A_1+\Omega\otimes A_2+\cdots+\Omega^{2n-1}\otimes A_{2n-1})((\Omega^i)\otimes I)(R\otimes I)\\
&=H\Gamma L.
\end{aligned}$$

Remark 4. An easier version of Lemma 3 is valid for circulant matrices. More precisely, the matrix $H\Gamma L$ is symmetric, whenever, H and Γ are circulant matrices.

THEOREM 5. *There is a twin strongly regular graph,*

$$SRG(936, 375, 150, 150)$$

whose vertices can be partitioned into 26 *disjoint cocliques of size* 36.

Proof. Let H be any block negacyclic Bush-type Hadamard matrix of order 36 and $W = [w_{ij}]$ a symmetric $BGW(26, 25, 24)$ with zero diagonal over the cyclic group C_{12}. It is shown in [9] that $D = [Mw_{ij}]$, where $M = H - I_6 \otimes J_6$ is a twin 2-(936, 375, 150) design by splitting it into

$$D^+ = \tfrac{1}{2}[P\,|w_{ij}| + Mw_{ij}]$$

and

$$D^- = \tfrac{1}{2}[P\,|w_{ij}| - Mw_{ij}],$$

where $P = J_{36} - I_6 \otimes J_6$, and showing that D^+ and D^- are both symmetric 2-(936, 375, 150) designs. All we need to do now is to multiply each block of D from the right by the matrix $L = R_6 \otimes I_6$ to get the matrix $D' = [Mw_{ij}L]$. The matrix D' is:

- symmetric with zero diagonal blocks. This follows from Lemma 3 (note that the matrix M is negacyclic) and the fact that the diagonal entries of the matrix W are all zero.
- a twin $SRG(936, 375, 150, 150)$. To see this, note that the symmetric matrix $D'^+ = \tfrac{1}{2}[(P\,|w_{ij}| + Mw_{ij})\,L]$ is a 2-(936, 375, 150) design (we need the remark after Lemma 3 to show that the matrix $[P\,|w_{ij}|\,L]$ is symmetric here). The same is true for the symmetric matrix $D'^- = \tfrac{1}{2}[(P\,|w_{ij}| - Mw_{ij})\,L]$.

THEOREM 6. *There is a Siamese twin strongly regular graph,*

$$SRG(1800, 1029, 588, 588),$$

whose vertices can be partitioned in 50 *cocliques of size* 36. *The two graphs share only cliques of size* 6.

Proof. Let H be any block negacyclic Bush-type Hadamard matrix of order 36 and $W = [w_{ij}]$ a symmetric $BGW(50, 49, 48)$ with zero diagonal over the cyclic group C_{12}. Let $M = H - I_6 \otimes J_6$. It is proven in [10] that

$$\tfrac{1}{2}[Mw_{ij} + ((J_6 + I_6) \otimes J_6)\,|w_{ij}|]$$

and

$$\tfrac{1}{2}\left[-Mw_{ij}+((J_6+I_6)\otimes J_6)\,|w_{ij}|\right]$$

are both symmetric 2-(1800, 1029, 588) designs. The fact that these two matrices have their diagonal blocks zero is a consequence of the fact that the $BGW(50, 49, 48)$ has all its diagonal entries zero. We now need to make these two matrices symmetric and we can do this by multiplying all the blocks of each of the two matrices from the right by the matrix $L = R_6 \otimes I_6$ of Lemma 3. This gives us a Siamese twin $SRG(1800, 1029, 588, 588)$ whose vertices can be partitioned into 50 disjoint cocliques of size 36, sharing only the cliques of size 6.

Remark 7.

- Clearly the construction method above works for any block negacyclic Bush-type Hadamard matrix and a symmetric BGW with zero diagonal of suitable parameters.

- It should be now clear for the reader the reason that a block negacyclic Bush-type is necessary in our construction above. Of course, it would be nice if one could use objects with less complexity in their structures than the block negacyclic Bush-type Hadamard matrices. But this may not be that easy. Indeed, there seems to be a balance in the complexity of the structure of the main ingredients for our construction. On one hand it is fairly easy to construct block negacyclic Bush-type Hadamard matrices for all orders $16n^2$ for which there is a Hadamard matrix of order $4n$. But then it is very hard to construct a proper symmetric BGW. On the other hand it is very hard to construct block negacyclic Bush-type Hadamard matrices of order $4n^2$, n odd, $n > 1$, whereas the required BGW is fairly simple to construct.

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REFERENCES

1. T. Beth, D. Jungnickel, and H. Lenz, "Design Theory," 2nd ed., Cambridge Univ. Press, Cambridge, UK, 1999.
2. G. Berman, Families of skew-circulant weighing matrices, *Ars Combin.* **4** (1977), 293–307.
3. A. E. Brouwer, Strongly regular graphs, *in* "The CRC Handbook of Combinatorial Designs" (C. J. Colbourn and J. H. Dinitz, Eds.), pp. 667–683, CRC Press, Boca Raton, FL, 1996.

4. K. A. Bush, Unbalanced Hadamard matrices and finite projective planes of even order, *J. Combin. Theory* **11** (1971), 38–44.
5. P. Delsarte, J. M. Goethals, and J. J. Seidel, Orthogonal matrices with zero diagonal, II, *Canad. J. Math.* **23** (1971), 816–832.
6. P. Gibbons and R. Mathon, Construction methods for Bhaskar Rao and related designs, *J. Austral. Math. Soc. Ser. A* **42** (1987), 5–30.
7. D. R. Hughes and F. C. Piper, "Design Theory," Cambridge Univ. Press, Cambridge, UK, 1985.
8. D. Jungnickel and V. Tonchev, Perfect codes and balanced weighing matrices, *Finite Fields Appl.* **5** (1999), 294–300.
9. H. Kharaghani, On the twin designs with the Ionin–type parameters, *Electron. J. Combin.* **7** (2000), R1.
10. H. Kharaghani, On the Siamese twin designs, *in* "Finite Fields and Applications" (D. Jungnickel and H. Niederreiter, Eds.), pp. 301–312, Springer-Verlag, New York/Berlin, 2001.
11. H. Kharaghani, New classes of weighing matrices, *Ars Combin.* **19** (1985), 69–72.